Numerical Solution of One-Dimensional Stefan-Like Problems Using Three Time-Level Method

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Abstract: We introduce formulation based on three time-level finite-difference method for one-dimensional Stefan-like problems. Matrix method stability analysis of equations is also investigated. The computational results obtained by the present method are in excellent agreement with the results reported previously by the other authors.

Keywords: numerical solution, three time-level method

INTRODUCTION

Problems in which the solution of a differential equation has to satisfy certain conditions on the boundary of a prescribed domain are referred to as boundary-value problems. In many important cases, however, the boundary of the domain is not known in progress but has to be assigned as part of solution. The term free-boundary problem is frequently used when the boundary is stationary and a steady-state problem exists. Moving boundary, on the other hand, is connected with time-dependent problems and the position of the boundary has to be described as a function of time and space.

In all cases, two conditions are wanted on the free or moving boundary, one to describe the boundary itself and the other to mature the definition of the solution of the differential equation. Convenient conditions on the fixed boundaries and, where favorable, an initial condition is also prescribed as usual. Moving boundary problems are often called Stefan problems, with reference to the early work of J. Stefan who, around 1890, was interested in the melting of the polar ice cap Crank (1987).

Many transient problems arising in science, engineering and industry involve a domain whose boundary changes its shape and size in time. Such problems have become well known as moving boundary problems or free boundary problems and such problems are often referred to as Stefan problems after Stefan, see Crank (1987), Furzeland (1977), Hill (1987). These problems are naturally nonlinear since the moving boundary is not known a priori and must be determined as a part of the solution. Numerical treatments of Stefan problems have been commonly based on finite-difference and finite-elements. Owing to do the nonlinearity of the moving interface, it is difficult obtain analytical solutions of these type problems excepting a restricted number of special cases Hill and Carslaw (1987, 1959). A perfect examine of these techniques and other various methods can be found in Crank (1957), Crank & Öziş (1980), Öziş & Gülkaç (2003), Gülkaç & Öziş (2004), Gülkaç (2005), Radhey et al (1985), Crank & Gupta (1975), Sparrow & Hsu (1981), Smith (1987), Landau (1950), Furzeland (1980), Fasano & Primicerio (1979), Muehlbauer (1965).

In this paper, three time-level method based on the implicit finite-difference are applied to the one-dimensional Stefan-like problems. The computational results obtained by the present method are in excellent agreement with the results reported previously by other researchers.
**Problem Description**

We consider a Stefan problem for which the exact solutions has been obtained by Cho (2002). Referring to Kutluay (2005), U(x,t) is the phase-change temperature and s(t) is the position of the moving boundary, a dimensionless model of the one-phase Stefan Problem with a forcing term in a semi-infinite slab of material that is initially at its melt temperature is as follows:

The temperature U(x, t) satisfies the heat conduction equation

\[
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + f(x,t), \quad 0 < x < s(t), \quad t > 0
\]  

subject to the boundary and initial conditions.

\[
U(0, t) = 0, \quad t > 0 \tag{2}
\]

\[
U(s(t), t) = 0, \quad t > 0 \tag{3}
\]

\[
U(x,0) = g(x), \quad 0 \leq x \leq 1 \tag{4}
\]

where \(f(x, t)\) and \(g(x)\) are described functions which are sufficiently smooth and nonnegative.

The moving interface s(t) satisfies the energy balance equation known as Stefan condition

\[
\frac{ds}{dt} = -\frac{\partial U}{\partial x}, \quad x = s(t), \quad t > 0 \tag{5}
\]

subject to \(s(t)=1, \; t=0\).

**Problem a:** Referring to Kutluay (2005), for this problem we first consider
\[f(x, t) = x \exp(t)+2\]
\[g(x) = x (1-x)\]
referred to Muehlbauer (1965), this problem has the following exact solution for \(U(x,t)\) and \(s(t)\), respectively:

\[
U(x, t) = x (\exp(t)-x), \quad 0 \leq x \leq s(t) \tag{6}
\]

\[
s(t)=\exp(t), \quad t \geq 0. \tag{7}
\]

**Problem b:** In this problem, we consider

\[
f(x, t) = -2 \frac{\dot{h}(t)}{h(t)} + 6x \frac{\dot{h}(t)}{h^2(t)} + x^2 \left( \frac{\ddot{h}(t)}{h(t)} - \frac{\dot{h}^2(t)}{h^2(t)} \right) + x^3 \left( 2 \frac{\dot{h}^2(t)}{h^3(t)} - \frac{\dot{h}(t)}{h^2(t)} \right)
\]

\[
g(x) = 0.5 \pi x^2 (1-x)
\]

where \(h(t) = 1 + 0.5 \sin(\pi t)\). Referring to Cho (2002) this problem has the following exact solution for \(U(x,t)\) and \(s(t)\), respectively:

\[
U(x, t) = \frac{0.5 \pi x^2 \cos(\pi t)(1+0.5\sin(\pi t)-x)}{(1+0.5\sin(\pi t))^2}, \quad 0 \leq x \leq s(t) \tag{8}
\]

\[
s(t) = 1 + 0.5 \sin(\pi t), \quad t \geq 0. \tag{9}
\]
Three time level finite-difference regularization

By using the three time-level finite-difference, we can get the equation (1):

\[
\frac{1}{12k} \left\{ \frac{3}{2} (u_{i+1,j+1} - u_{i+1,j}) - \frac{1}{2} (u_{i,j+1} - u_{i+1,j-1}) \right\} + \frac{5}{6k} \left\{ \frac{3}{2} (u_{i,j+1} - u_{i,j}) - \frac{1}{2} (u_{i,j} - u_{i,j-1}) \right\} \\
+ \frac{1}{12k} \left\{ \frac{3}{2} (u_{i-1,j+1} - u_{i-1,j}) - \frac{1}{2} (u_{i-1,j} - u_{i-1,j-1}) \right\} - \frac{1}{h^2} (u_{i+1,j+1} - 2u_{i,j} + u_{i-1,j+1}) = f(x,t)
\]

(6)

and truncation error of equation (6) is \( O(k^2) + O(h^4) \). And equation (6) can be written as

\[
\left( \frac{3}{2} - 12r \right) u_{i-1,j+1} + (15 + 24r) u_{i,j+1} + \left( \frac{3}{2} - 12r \right) u_{i+1,j+1} = 2u_{i-1,j} + 20u_{i,j} + 2u_{i+1,j} - \frac{1}{2} u_{i-1,j-1} \\
- 5u_{i,j-1} - \frac{1}{2} u_{i+1,j-1} + 12k f(x_{i,j}, t_j)
\]

(7)

where \( r = \frac{k}{h^2} \).

For known boundary values they can be written as

\( u_{0,j} = 0 \)
\( u_{N,j} = 0 \)

Hence, the equation (7) can be written matrix form as:

\[
Au_{j+1} = Bu_j - Cu_{j-1} + D_j
\]

(8)

\[
u_{j+1} = A^{-1}Bu_j - A^{-1}Cu_{j-1} + A^{-1}D
\]

(9)

where \( D \) is a vector of known \( f(x,t) \) values and

\[
A = \begin{bmatrix}
15 + 24r & \frac{3}{2} - 12r \\
\frac{3}{2} - 12r & 15 + 24r & \frac{3}{2} - 12r \\
& \ddots & \ddots & \ddots \\
& & \frac{3}{2} - 12r & 15 + 24r
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
20 & 2 \\
2 & 20 & 2 \\
& & \\
& & \\
2 & 20 \\
\end{bmatrix}
\quad \text{and} \quad
C = \begin{bmatrix}
5 & 1 & 1 \\
1 & 5 & 1 \\
1 & 1 & 5 \\
\end{bmatrix}
\]
are of order (N-1).

Equation (9) can be written as

\[
\begin{bmatrix}
u_{j+1} \\
u_j \\
u_{j-1} \\
\end{bmatrix}
= \begin{bmatrix}
A^{-1}B & -A^{-1}C \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
u_j \\
0 \\
u_{j-1} \\
\end{bmatrix}
+ \begin{bmatrix}
A^{-1}D \\
0 \\
0 \\
\end{bmatrix}
\]
i.e. \( v_{j+1} = P v_j + C \). A has distinct eigenvalues and all the sub matrices of P commute with each other so

the eigenvalues \( \lambda \) of P are the eigenvalues of

\[
\begin{bmatrix}
\xi_k & -\xi_k \\
\mu_k & \mu_k \\
\end{bmatrix}
\]
where \( \xi_k \) is the kth eigenvalue of B, \( \xi_k \) is the kth eigenvalue of C and \( \mu_k \) is the kth eigenvalue of A. As

\[
\det(P - \lambda I) = 0 = \lambda^2 - \left( \frac{\xi_k}{\mu_k} \right) \lambda + \left( \frac{\xi_k}{\mu_k} \right)
\]

where \( \xi_k = 20 + 4 \cos \frac{s\pi}{N+1} \), \( \xi_k = 5 + \cos \frac{s\pi}{N+1} \) and \( \mu_k = 15 + 24r + (3 - 24r) \cos \frac{s\pi}{N+1} \)

and \( \lambda = \left\{ \xi_k \pm \sqrt{\xi_k^2 - 4\xi_k \mu_k} \right\} / 2\mu_k \).

Hence, \(|\lambda| \leq 1\) for all \( r \). Therefore the equations are unconditionally stable. It is shown in Richtmyer & Morton (1967) that same equations are also stable for all \( r>0 \) as mesh lengths tend to zero.

**CONCLUSION**

In this paper, we presented an implicit finite-difference method for one dimensional Stefan-like problem. All the computational results obtained by using the schemes outlined in the previous section show good agreement with the exact solution. As a conclusion the three time-level finite-difference method is accurate and efficient to solve Stefan-like problems. Note that the results obtained by the present method are excellent agreement with the exact results of Cho (2002) and are superior to the results obtained by Kutluay (2005).

Table 1. shows the values of the temperature distribution of problem (a) for the indicated values of \( t \) obtained by the present method, along with the corresponding results of the other authors. Table 2. shows the values of the temperature distribution of problem (b) for the indicated values of \( t \) obtained by the present method, along with the corresponding results of the other authors.
Table 1. Values of the temperature distribution of problem (a) at \( t_f = 1.0 \), \( h=0.0125 \). Values of the interface movement at \( t_f = 1.0 \) 2.718280, exact solution 2.718282.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Numerical solution ( h=0.005 )</th>
<th>Exact solution [18]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present Kutluay [19]</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.665024 0.665109</td>
<td>0.665015</td>
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<tr>
<td>0.2</td>
<td>1.182251 1.182432</td>
<td>1.182249</td>
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<td>1.551709 1.551959</td>
<td>1.551702</td>
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<td>1.773373</td>
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<td>0.5</td>
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<td>1.847264</td>
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<tr>
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<td>1.773373</td>
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<tr>
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<td>1.551705 1.552004</td>
<td>1.551702</td>
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<tr>
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<td>1.182249 1.182485</td>
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</tr>
<tr>
<td>0.9</td>
<td>0.665015 0.665150</td>
<td>0.665015</td>
</tr>
</tbody>
</table>

Table 2. Values of the temperature distribution of problem (b) at \( t_f = 1.0 \), \( h=0.0125 \). Values of the interface movement at \( t_f = 1.0 \) 1.475520, exact solution 1.475528.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Numerical solution ( h=0.005 )</th>
<th>Exact solution [18]</th>
</tr>
</thead>
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<tr>
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<td>0.091670 0.091390</td>
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</tr>
<tr>
<td>0.9</td>
<td>0.058009 0.057829</td>
<td>0.058014</td>
</tr>
</tbody>
</table>

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REFERENCES

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